



This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

- + *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.
- + *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.
- + *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.
- + *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at <http://books.google.com/>



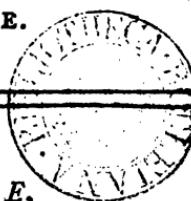


10. 2. - 6

5-7-181

A
L E T T E R
TO
The Reverend Dr. *POWELL*,
Fellow of St. John's College, *Cambridge*;
IN ANSWER TO HIS
OBSERVATIONS on the First Chapter
Of a BOOK called
MISCELLANEA ANALYTICA,
AND HIS
DEFENCE of those OBSERVATIONS.

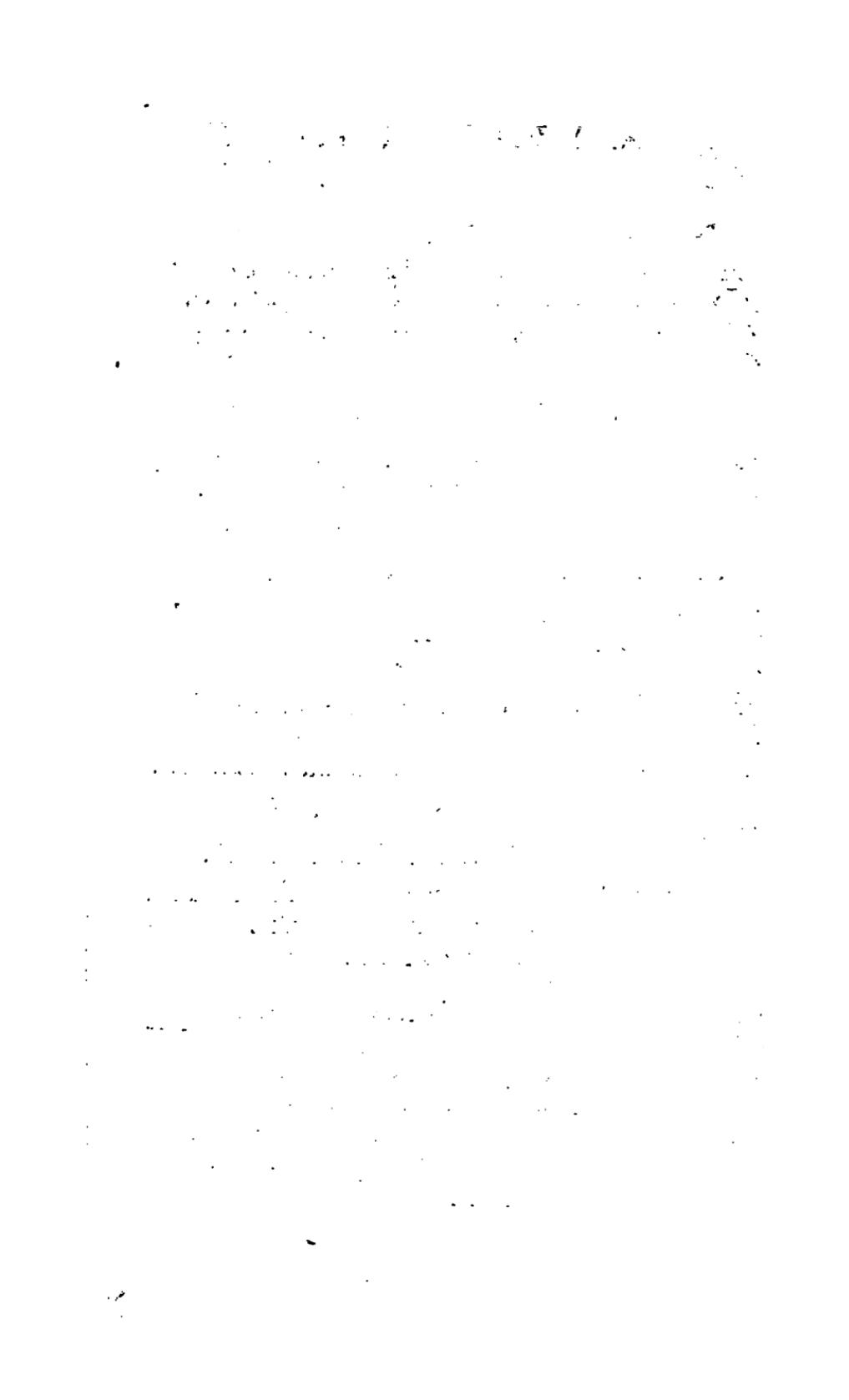
BY
EDWARD WARING, M.A.
LUCASIAN PROFESSOR of MATHEMATICS,
and Fellow of MAGDALEN COLLEGE,
CAMBRIDGE.



CAMBRIDGE,
Printed by J. BENTHAM, Printer to the UNIVERSITY,
for W. THURLBOURN & J. WOODYER in *Cambridge*.

M.DCC.LX.

181. f. 56.



A LETTER, &c.

Rev^d SIR,

AS you are confessedly the Author, both of the *Observations* on the first chapter of the *Miscellanea Analytica*, and of the *Defence* of those *Observations*; I have therefore addressed this letter publickly to you, because I can see no reason, why my Name should be expos'd, whilst Your's is a secret; or to give you your own illiberal metaphor back again, I can see no reason, why you should be suffered to lye skulking under the table, *whilst I am playing above board.**

You would have it believed, that it was a love for the Science and the University, which induced you to write and publish, those *Observations*.† But a lover of the Science, would not have begun as you do, with that impertinent reflexion on the intricacies of modern *Algebra*; and a lover of the University, would not have persecuted with so much cruelty, *a plain modest Man*, ‡ at least a harmless Member of that University.

But if this reason, which you assign for writing and publishing your *Observations* be improbable, the reason, which you have given for still *defending* them, is plainly ridiculous. You have it seems *enough of Philosophy*, to bear patiently that scorn, and contempt, with which you pretend to be treated in the *Reply*; but I have, you say, inconsistently enough “added complaints to contempt; crying out “of one, whom I trample beneath my feet, as *unkind, severe, and cruel.*” And these are the accusations,

A 2

to

* Def. p. 5. † Obs. p. 3 & 4.

‡ The Professor, a plain modest man. Def. p. 18.

to which (such is the tenderness of your nature,) "you cannot willingly submit." Now the *Reply* is generally acknowledged to be written with that plainness and simplicity, which becomes the subject. Certainly there is no expression in it, that was intended to give you offence; nor can I, upon the most careful perusal of a hasty performance, find any, that in my judgment, could reasonably offend you: nor have you, either here, or in any other part of your work, quoted so much as a single passage, in which you are treated even with incivility or disrespect: much less have you proved, what you so roundly assert, that I scorn and despise you, and even *trample you beneath my feet*.

But whilst you were thus complaining of indignities you had never suffered, did not your recollection once call to your mind, that haughty disdain, and those cruel mockings, with which you had really trampled upon the neck of your poor young *Analyst*? He is your *Writer*, — your *Mathematician*, — your *Calculator*, — your *Analyst*, — your young *Analyst*, — your *proceder according to the famous Binomial Theorem*, with that contemptible parenthesis, "so far as he understands it, I suppose." And what reason then have you to suppose, that he does not understand it? I suppose, because he did not, as you did, use a double application of that immense theorem, to find out a quotient of four terms only, of a plain easy sum in common division.*

You have another parenthesis of the same cast, in the 16th page of your *Observations*. "It is not 'our Author's practice, to demonstrate his propositions; he sometimes tells us whence (AS HE SUPPOSES) demonstrations of them may be drawn."

Now

Now he gave you a demonstration of his first lemma, and you mistook his demonstration for a solution : he demonstrated his second lemma too, and you mistook his demonstration, for a number of rules, by which the lemma might be adapted to each particular case.* And lastly, he gave you a solution of the problem contained in the second corollary to this lemma ; and “ with so little attention did you read the book, you was writing against, that you mistook some steps in the investigation, for the answer to the problem.”†

Again, you have another of these parentheses in pag. 13. “ according to the rules (which he has “ learned)” — “ and follows blindly,” as you interpret it yourself, in the *Defence*, though it seemed to want no comment.

And it is upon this occasion, that contrary to all the rules of good writing, you introduce the doctrine of infinites, where there is no sort of necessity for it. And though you was unable to carry it beyond the very first step ; and not even so far, without a most egregious blunder (it is your own word) ; yet having discovered that *mysterious* quotient, which I have mentioned above ; and which, when you had found it, you did not know to be a quotient ; you fall into a fit of self-adulation, and admiring the wonders, which you have worked in these deeps of infinity, you set forth the **EXTENSIVE USE**, and **INCOMPARABLE ELEGANCE** of that doctrine, when under the management of a **MASTER** ; and how it must often betray such a fool as you conceive me to be, *into inextricable difficulties.*‡

But this *parenthesis* seems to be a favourite figure with

* *Rep.* p. 13. † *Rep.* p. 19. ‡ *Obs.* p. 14, 15, and
Rep. p. 24 and 25.

with you; for besides those which I have already mentioned, you have several others of a like illiberality.

The same answer will do for all of them; which is, that an author wants but one talent to be able to say such things as these are; and this talent is not modesty certainly; nor yet learning; nor wit neither. And therefore they may be, and often have been said of the best writers, by the worst: and for this reason it can be no disgrace to me, to have been the Object of such abuse; no more than it can be a credit to a person of your degree, and station in the University, to have been the Author of it.

The rest of your performance is of a piece with what I have already quoted from it. It is in no respect, what you promised in the beginning, a candid enquiry into the merits, as well as the faults of my book; but it is a domineering, cavilling, malicious declamation: utterly repugnant to the nature of your subject, the language of which can have no other beauty, but that of plainness and perspicuity: and no less inconsistent with that singleness of heart, and simplicity of expression, which should distinguish the character you assume, of *a lover of the Science and the University*.

Your objection to the first corollary to my second lemma, is, the *uselessness* of it. But your proof? Why, I have transcribed an example of it from *Newton*. What follows? That I could find no other example. It had been well, if you had given us another of your rules of logic, by which we might have connected these two propositions. But indeed it is an argument which a fair Observer would have scorned even to mention; and yet it seemed to you to deserve, all the ornaments of

of your rhetorick, and to be preceded by a long train of an impertinent

INTRODUCTION.

“ A common man would think, that, when he had been taught to find each of these quantities, he should not want a series to help him to add them together. But in these enquiries, we do not attend only to the use of a proposition, but to its extent, regularity, and easy connexion, with other parts of the subject. Great writers are indeed often able to unite use and beauty, in others we may be content to meet with one of them. Concerning the ease and elegance of the rule before us, I will give no judgment.”*

ARGUMENT.

“ But how little use could be made of it, even by its Author, may be easily conjectured; since when *all IMAGINABLE* equations *lay OPEN* before him, he has chosen for his example, that one which was already computed for him in the *Arithmetica Universalis.*”†

I had answered to this argument, that, “ though you seemed to imagine, that I was afraid to trust my own solution, if you would do me the honour to give me any equation, not made difficult on purpose, I would engage to write down the answer, as fast as I could write with any care and attention.” You reply, “ that I should sadly waste my ink and paper,” where I think we may leave this argument.

Your objection to my third lemma, is the obscurity of it.

“ He must have more skill, than the honest study of mathematicks will ever furnish, who

* Obs. p.11.

† Obs. p.11,12..

“ is able to discover the *bidden treasures*, of the
“ third lemma, from any *marks* on its *surface*.” †

But if we examine this allegory coolly, as becomes two mathematicians, we shall find,

1st, That it does no credit to your judgement ; because to talk of the **HIDDEN TREASURES** of a sum in arithmetic, and the *Black Art*, that is necessary to discover them ; this colouring is by much too high for the plain subject of your picture.

2dly, It does no more credit to your fancy, than your judgement ; because it is a stale hackneyed allegory ; and it was your memory that supplyed you with it, and not your fancy.

3dly, It does still less credit to your sincerity ; because as soon as the force of your allegory has spent itself, you confess that an honest man may come at the meaning of the problem, without dealing with the devil. “ Neither the words in which “ it is proposed, nor the rules given for the solution, “ will conduct us to them (i. e. *bidden treasures*) : “ But the row of symbols at the bottom of the “ page, together with the two examples that follow it, may perhaps point out the design of the “ problem.”

The problem is certainly difficult, and you say, that in order to explain it, I have given,

1st, A general expression of it.

2dly, Some rules for the solution.

3dly, A row of symbols.

4thly, An example, and

5thly, Another example ; and
surely this account does no discredit to my industry.

Your objection to the 1st corollary to the 2d lemma, is of the same nature.

The

The words you say, *may puzzle an attentive reader*; but if the series which follows be examined, the design of the corollary will appear; and in your defence, you have been so disingenuous, as to quote the general expression without the series to explain it. You might in the same manner prove *Euclid* to be an obscure writer, since there are propositions in the elements, *whereof the words, in which they are expressed, might puzzle an attentive reader*; and yet if the examples which follow be examined, the design of them will appear.

But to return to the lemma.

You want two exceptions to it,

1st, You would have different powers of the same root, and

2dly, The same power of different roots, excepted out of the problem.

Your words are, "except those combinations which contain two or more powers of the same roots, or the same power of two or more roots."*

I had answered, that if I understood you right, the exceptions, which you wanted, were mention'd in the lemma †. You reply, it is wonderful, if these exceptions are mentioned, that he did not transcribe the two or three words which contain them"‡. But it is more wonderful, that you should still insist upon this, when the words which contain these exceptions have been already transcribed by Mr. Wilson. The problem, says he, is "To find the aggregate of all the different combinations that can be made of the m^{th} power of *one root*, into the p^{th} of *another, &c*". Where it may be observed by the way, that the first exception is evidently

B im-

* *Obs.* p. 16. † *Rep.* p. 28. ‡ *Def.* p. 34.

implied in the lemma itself. § The latin is “ *in p
potestatem ex quâque aliâ.* ” *It should have been
added*, I suppose, *et non eâdem*. But Mr. Wilson
goes on, “ The 2d exception taken notice of in the
“ Observations, is made by the Author himself, in
“ these words immediately following the example.”

Cum duæ vel tres vel plures potestates sunt æqua-
les, dividatur summa per producta 1. 2. 3. &c. ut
in præcedente lemmate.

And yet in your Observations, you flatly assert,
that not a hint is given of these exceptions in my
book : * and you still maintain this assertion in
your Defence.

But to return to your rhetoric.

“ We will not insist on method, if we can
“ come at truth. Some patience will be necessary:
“ the WELL, in which it is here concealed, is *very*
“ *deep.* ” * Upon which *deep well*, what I have
before said of *hidden treasures*, might be repeated.
But since you are continually returning to this
charge of *obscurity*, give me leave to tell you
plainly, that though I were as *obscure* as you re-
present me to be, yet you are not a writer to
throw a stone at me. For if I am *obscure*, you are
impenetrable. One example will be enough to
prove it, and here it stands.

“ In the next corollary it is proposed to find the
“ aggregate of the sum of the roots, of their squares,
“ of their cubes, &c. in infinitum, it should have
“ been added, if each root of the equation be less
“ than 1 ; and greater than — 1. It is easy to see in-
“ to what Absurdities the reasoning here used will
“ lead us ; suppose for example that one of the
roots

§ Wilson’s Vindication, p. 13. * Obs. p. 16. † Obs. p. 21.

“ roots of the equation should be 4: then by the
“ rule before $1+4+16, \&c. = \frac{1}{3}$. ”

You say that the problem is to find the aggregate of the sum of the roots of any given equation, of their squares, of their cubes, &c. ad infinitum: and that supposing 4 to be one of the roots of the equation, then by the rule before you $1+4+16+64, \&c. = \frac{1}{3}$; which I say is utterly unintelligible. If you had said, then $4+16+64, \&c. = \frac{4}{3}$, the reader would not have been quite in the dark; because 4, 16, 64, &c. are the powers of (4), which is supposed to be one of the roots of the equation. But I defy the most partial of your friends to say, that he even guessed at the meaning of this (1), which leads the series; since it is certainly neither 4, nor the square of 4, nor any other power of 4.

You was writing against an *unpublished* book, and therefore your pamphlet should have been intelligible of itself; because it was not probable, that your reader would have my book to refer to. But suppose him to have had the book before him; he must still most likely have been not a little puzzled, since your series, as is observed in my *Reply*, was not made from the rule itself, which you quote for it; but from one of the steps, which is made use of in the investigation of the rule. There are four steps in the solution of the problem of this form $\frac{1}{3} = 1+A+A^2+A^3, \&c.$, substitute (4) for A , and you have $\frac{1}{3} = 1+4+16+64, \&c.$ and this is your series. And thus you refer to the rule, and expect your reader should consult the investigation.

But your defence of this passage is pleasant enough. You don't deny the fact, that you made

your series from these steps in the investigation, instead of making it from the rule itself. It was impossible for you to deny the fact; but after abundance of words, you tell me, that an *error in the investigation of a rule, must be an error in the rule itself*; * and that you had in your pocket, a quadratic, a cubic, and a biquadratic equation; by either of which you can prove, from my rule, that $\frac{1}{3} = 1 + 4 + 16 + 64$, &c. Not to mention that a simple equation would have done your business, (because you despise simplicity) you must tell me, by what *honest skill* in the science, your reader was to *discover* these *bidden equational treasures*, at the bottom of your pocket. The *well* is *very deep*, and, I suppose, not very accessible.

In your *Observations*, this series was intended for a *familiar* illustration, of what you had truly asserted; that if either of the roots of the equation proposed, be greater than (1), the problem admits no answer: and you introduce it as such: "It is easy to see, &c." But, in your *Defence*, you have made a formal proposition of it. "That the Rule here given *contains* this, among innumerable other absurdities, that $1 + 4 + 16 + 64$, &c. $= \frac{1}{3}$;" and you have proved it by three long pages of calculation.

But who had denied, that this absurdity was *contained* in the rule, supposing either of the roots might be greater than (1)? I am sure, I had not: I had only asserted, what I still assert, and what you cannot deny, that you did not make your series from the rule itself, to which you refer your reader; but from one of the steps in the investigation,

* *Def.* p. 26.

tion, which you blundered upon instead of the rule.

You say very truly, that *it is easy to see*; and I add, that it was as easy to shew, without three pages of calculation, that it would be absurd to suppose the problem to extend to such cases, where either of the roots are greater than (1).

Let (4) be the root of a simple equation. Then the problem is to find the sum of $4 + 16 + 64$, &c. ad infinitum; which sum is plainly infinite, and the very terms of the problem imply an impossibility: and this is the great mystery, concerning which you *reasonably hope, that I and my assistant in the Reply, will never again assert, that you did not understand it, or that we did.* But what I asserted before, I must still assert, That though you saw this absurdity, yet you did not shew it. That you made your series not from the rule but from the steps. That to the bulk of your readers, who wanted the book to refer to, the passage, which I have quoted, must have been total darkness. That of the few, who had the book, I may well venture to defy you to produce a single man that understood it. And lastly, I say, that the proof you have now produced, is awkward and cumbersome; and that to introduce three long pages of calculation, where no calculation was required, shews plainly, that you yourself are deficient in that *Ease* and *incomparable Elegance* in writing, the want of which in my book, you so often complain of.

But enough has been said of the manner, in which your *Observations* are written, to convince the reader of the truth of what I have asserted, that your *Pamphlet* is no serious, impartial enquiry.

ry; but a verbose, virulent declamation: from one end to the other of which you treat your *Mathematician* with a most sovereign contempt, and formally condemn his book to the flames, in the conclusion. And yet the charge of cruelty is an *accusation to which you cannot willingly submit.**

But you may find it difficult to exculpate yourself; for if the manner, in which your pamphlet is written, be cruel; the occasion on which it was published, is still more so. To fall upon a *plain modest Man*, when his mind was distracted, with the uncertain expectation of an approaching, contested election, was surely, I do not say a most unchristian, but a most inhuman act of barbarity. And considering the anxiety of my situation, and that the election was not many days distant, did not you hope that your *Observations* would have done their business, before they could possibly be contradicted?

My *Reply* however was published before the Election, and I had the satisfaction to hear it well spoken of every where; even your own friends seemed not unwilling to acknowledge, that the pamphlet had some merit in it: but they soon began to insinuate, that this merit was none of mine, and, what no other person would have done without the most positive, and incontestible evidence, you, in your *Defence*, without any evidence at all, have plainly attributed it to another: thereby endeavouring, not only to deprive me of the little praise I had gained by the *Reply*; but at the same time to make a Gentleman a party in the dispute, who was no otherwise concerned in it, than having spoke his sentiments with that

free-

* *Def.* p. i.

freedom, with which all men have a right to speak, and generous men think themselves obliged to speak on such occasions.

That my *Reply* underwent the chastisement of my friends, I do not deny: and if your *Observations* had undergone the same discipline, I am inclined to think so well of your friends, as to believe, that this dispute would never have arisen between us.

And so much for this subject. I proceed to

LEMMA I.

TO find the dimensions of an equation, by which any proposed equation ($x^n - Ax^{n-1} + Bx^{n-2}, \&c. = 0$) of (n) dimensions; may be reduced to one of (m) dimensions; such as ($x^n - ax^{n-1} + bx^{n-2}, \&c. = 0$) supposing (n) to be greater than (m), and that the roots of the *reduced* equation $x^n - ax^{n-1} + bx^{n-2}, \&c. = 0$, be roots likewise of the equation $x^n - Ax^{n-1} + Bx^{n-2}, \&c. = 0$, from whence it was reduced.

My answer is, that the dimensions of the *reducing* equation will be equal to the fraction $\frac{n, n-1, n-2, n-3, \&c.}{1, 2, 3, 4, \&c.}$ both numerator and denominator being continued to as many terms, as there are units in (m).

Suppose, for example, the question was, to find a quadratic equation, such as $x^2 - ax + b = 0$, whose two roots should be two of the roots of any given biquadratic; such as $x^4 - Ax^3 + Bx^2 - Cx + D = 0$; where A, B, C, D , the coefficients of the given equation, are supposed to be known; a and b , the coefficients of the equation sought, unknown.

Now

Now what the lemma requires is, not actually to solve the problem, but to determine the dimensions of it; not to find a and b , the coefficients of the equation sought, but to shew, before we enter upon the investigation, how high the equation will rise, by which the values of these coefficients are to be found. And in order to this, I say in the first place; that the equation $x^4 - Ax^3 + Bx^2 - Cx + D = 0$, being of four dimensions, has four roots: suppose the roots of it therefore to be $\alpha, \beta, \gamma, \delta$, and with every pair of these roots, we may form a quadratic equation, that will answer the conditions of the problem.*

$$\text{Such as } x^2 - \frac{\alpha}{\beta}x + \alpha\beta = 0 \quad \alpha \text{ and } \beta$$

$$x^2 - \frac{\alpha}{\gamma}x + \alpha\gamma = 0 \quad \alpha \text{ and } \gamma$$

$$x^2 - \frac{\alpha}{\delta}x + \alpha\delta = 0 \quad \alpha \text{ and } \delta$$

$$x^2 - \frac{\beta}{\gamma}x + \beta\gamma = 0 \quad \beta \text{ and } \gamma$$

$$x^2 - \frac{\beta}{\delta}x + \beta\delta = 0 \quad \beta \text{ and } \delta$$

$$x^2 - \frac{\gamma}{\delta}x + \gamma\delta = 0. \quad \gamma \text{ and } \delta.$$

of which the roots are

There are therefore as many answers to the problem, as there are ways of taking the (4) roots of the proposed biquadratic by pairs: from whence I conclude, that the dimensions of the equation, by which the problem is answered, will be equal to the number of combinations of 4 things, taken two and two, which is $\frac{4 \cdot 3}{2} = 6$. †

But

* *Saund.* p. 185. † *De Moiv.* *Miscel. Anal.* p. 180.

But this seems to you a giant's stride, which nevertheless is the first step, that Sir Isaac Newton teaches us in his discourse *De Naturâ Æquationum*: where, after having defined the word *root*, and given an example of an equation with four roots, he adds, " et ne mireris eandem æquationem habere posse plures radices, sciendum est plures esse posse solutiones ejusdem problematis.

" Ut si circulorum duorum datorum quæreretur intersectio, duæ sunt eorum intersectiones, atque adeo quæstio admittit duo responsa, & perinde æquatio intersectionem determinans habebit duas radices, quibus intersectionem utramque determinet, si modo nihil in datis sit; quo responsum ad unam intersectionem determinetur." *

Nay, Dr. Saunderson too reasons in the same manner, in the very first of his problems producing quadratic equations.

But Sir Isaac Newton, having added another example to that I have already mentioned, concludes thus: " In omni igitur problemate necesse est æquationem, quâ respondetur, tot habere radices, quot sunt quæstæ quantitatis casus diversi ab iisdem datis pendentes, & eâdem argumentandi ratione determinandæ."

Now to apply this rule to the case before us. You may, if you please, begin your investigation, by assuming a quadratic, such as $x^2 - ax + b = 0$, and supposing the roots of it to be, for example, the two least roots of the proposed biquadratic. But this restriction would be impertinent, because it could have no influence on the process; there being no data in the problem, by which we may direct the investigation to any one of the six equations above mentioned, rather than any other.

Or to be still more distinct; as I have said before, to find the equation $x^4 - ax^2 + b = 0$, is to find the coefficients (a) and (b). Let us suppose then, that (a) the coefficient of the second term is first sought for: now it appears that (a) may have six values $\alpha + \beta, \alpha + \gamma, \alpha + \delta, \beta + \gamma, \beta + \delta, \gamma + \delta$, all depending on the same data: viz. that the two roots of the quadratic sought, are two of the roots of the proposed biquadratic; and to be discovered by the same method of arguing; since there is no circumstance in the problem, by which we may restrain the investigation, to one or more of these values of (a), exclusive of the rest. The equation therefore, by which the problem is answered, will have as many roots, as there are different values of (a); i. e. as there are different combinations of the four roots $\alpha, \beta, \gamma, \delta$ taken two and two, i. e. $\frac{4 \cdot 3}{2} = 6$; and consequently it will be of six dimensions: "Potest enim æqua-
" tio tot habere radices, quot sunt ejus dimensiones
" & NON PLURES." *

What has been said of the reduction of a biquadratic, to a quadratic, is, mutatis mutandis, applicable to all other reductions.

Suppose the given equation to be of 5 dimensions, $x^5 - Ax^4 + Bx^3 - Cx^2 + Dx - E = 0$; and let it be required to find a cubic, whose three roots shall be three of the roots of the given equation. Now an equation of 5 dimensions, has 5 roots; let therefore the 5 roots of this equation be $\alpha, \beta, \gamma, \delta, \epsilon$; and a cubic equation, whose three roots shall be any three of these roots, will answer the conditions of the problem.

Such

* *Newton. p. 242.*

$$-a + a\beta$$

Such as $x^3 - \beta x^2 + a\gamma x - a\beta\gamma = 0$, of
 $- \gamma + \beta\gamma$

which the Roots are α, β, γ .

$$-a + a\beta$$

Or $x^3 - \beta x^2 + a\delta x - a\beta\delta = 0$, of which the
 $- \delta + \beta\delta$

Roots are α, β, δ .

And so every combination of the five roots, taken three and three, will answer the conditions of the problem. From whence I infer, that the equation by which the problem is solved, must necessarily be of as many dimensions, as there are combinations of the five roots, taken three and three; (i.e.) $\frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3} = 10$.

And so in general, if the problem be, to find an equation $x^n - ax^{n-1} + bx^{n-2}, \text{ &c.} = 0$, whose (m) roots shall likewise be (m) of the roots, of any given equation, $x^n - Ax^{n-1} + Bx^{n-2}, \text{ &c.} = 0$, where (m) is supposed to be less than (n) ; the reasoning is still the same.

The equation $x^n - Ax^{n-1} + Bx^{n-2}, \text{ &c.} = 0$, has (n) roots: and out of any combination of (m) of these roots, may be formed an equation of (m) dimensions, that will answer the conditions of the problem: of which equations therefore, there will be as many, as there are combinations of the (n) roots, of the proposed equation taken (m) at a time: nor is there any thing in the problem, by which we may confine the investigation, to any one or more of these equations, exclusive of the rest; "omnes autem ab ipsis datis pendent, & eadem argumentatione determinanda sunt;"

“sunt:” from whence I conclude, that the equation by which the problem is solved, must have as many roots, and consequently be of as many dimensions, as there are combinations of (n) things, combined according to the number (m); that is, $\frac{n \cdot (n-1) \cdot (n-2) \cdots 1}{1 \cdot 2 \cdot 3 \cdots m}$ &c. both numerator and denominator, being continued to (m) terms.* And so much for the demonstration.

But you are not more offended at the brevity of my demonstration, than at the scarcity of my examples. “One only, you say, is mentioned in my book; and though the lemma is very extensive, no other is added in the Reply: †” which complaint would never have dropt from you, if you had been at all acquainted with the writings of *Des Cartes* and his commentators, of whom you speak so familiarly. I gave you one example from *Des Cartes* himself,‡ and you may find more in his commentators: consult *Hudde's* 19th rule, *de reduc. Aequat.* where you will find an equation of 5 dimensions, reduced to one of (2), by an equation of 10 dimensions; which is the number required by the lemma: since $10 = \frac{5 \cdot 4}{1 \cdot 2}$. In the same rule too, he reduces an equation of 6 dimensions, to one of (2), by an equation of 15 dimensions; which likewise is the number required by the lemma, since $15 = \frac{6 \cdot 5}{1 \cdot 2}$. To which he adds, that, by the same method, the equation of 6 dimensions, might be reduced to one of 3; but that the reducing equation would,

* *De Miro. Milel.* p. 180.

† *Def.* p. 7.

‡ *Saund. Alg.* p. 735.

in this case, rise to 20 dimensions ; now 20 =
the number required by the lemma. †

As to what you object, that there are particular equations, which may be reduced with more facility ; it is a mere cavil : since it was impossible for you to imagine, that I meant to assert the contrary, though I had said nothing about it. But I have expressly declared, in the 5th chapter, that in special cases, the reduction may be performed with less trouble ; which chapter you have quoted, disingenuously suppressing this declaration. Your rule of logic therefore, that “ a general affirmation is contradicted by a particular negation ; ” ‡ can only serve to shew the multiplicity of your learning ; since you know very well, that I never meant to say, that *no* equation could be reduced by an equation of fewer dimensions, than what the lemma requires. *Hudde* says, that if you would reduce an equation of (6) dimensions, to one of (3), the *reducing* equation will rise to 20 dimensions. Let therefore $x^6 = 1$, and by extracting the square root only, we shall have $x^3 = \pm 1$, without the unnecessary aid of an equation of twenty dimensions : and yet I am persuaded, that no reader will think this a serious objection to *Hudde*’s assertion. Sir *Isaac Newton* too asserts, that the *quinquisection* of an arc, is not to be performed without an equation of 5 dimensions ; and it is from this very example, that he deduces the rule upon which the lemma is founded : and yet every one knows, that *Euclid* has divided the circumference of a circle into 5 equal parts, without an equa-

† *Des Cart. Geo.* Amstell. p. 486.

‡ *Def.* p. 11.

equation of 5 dimensions.* So likewise it is a general rule, that the trisection of an arc, requires a cubic equation ; and yet *Euclid* has trifected the semicircumference, by a simple equation.

No candid reader, ever misunderstands such general expressions ; and the meaning of the lemma is obviously, that unless there be something peculiar in the equation to be reduced, the *reducing* equation will have as many dimensions, as the lemma requires.

But you object, that I myself have reduced equations of *five* and *six* dimensions, to others of *three*, by the intervention of equations of fewer dimensions, than the lemma requires. To which objection, however, I must beg leave to be silent, partly, because I have endeavoured to account for these solutions, in the place where they are given ; but principally, because my intention is to make this pamphlet as intelligible as I can : and to explain these solutions here, would take me off too long from my subject ; and to refer the reader to an *unpublished* book, would be as ridiculous in me, as it was ungenerous in you to write against it.

One instance however, of your disingenuous dealing on this subject must not be omitted. In your *Observations* you had objected to me my solutions of equations, of 4, 5 and 6 dimensions : but Mr. *Wilson* took the trouble to explain to you, my solution of biquadratics ; which therefore, in your Defence, you have prudently and silently dropped ; and to cover the artifice, you talk magnificently of my reduction of equations of 5 and

* *Eur.* B. 4. Prop. x.

6 dimensions to cubics; but concerning the bi-quadratic, you say nothing at all.

And thus much in defence of the lemma.

I proceed in the next place to take notice of some of the more notorious of the many mistakes, which you have committed in your animadversions upon it.

And first, you are egregiously mistaken in imagining, that the reduction of an equation of 6 dimensions (in which the alternate terms are wanting) to a cubic, is at all inconsistent with the Lemma.

Ex. Let $x^6 - 14x^4 + 49x^2 - 36 = 0$.
and supposing $x^2 = z$, and we have

$$z^3 - 14z^2 + 49z - 36 = 0.$$

Here you say we have an equation of 6 dimensions, reduced to one of three, by a simple quadratic $x^2 = z$; whereas by the lemma, the reducing equation ought to be of $\frac{6+4}{2} = 5$ dimensions. But indeed this reduction is so far from contradicting the lemma, that it has no relation to it; and you could not have fallen into this mistake, if you had at all regarded the conditions of the problem: one of which is, that the roots of the *reduced* equation, should be roots likewise of the equation, from which it is reduced.

Now the roots of the given equation

$$x^6 - 14x^4 + 49x^2 - 36 = 0,$$

are $+1, -1, +2, -2, +3, -3$.

But the roots of the reduced equation,

$z^3 - 14z^2 + 49z - 36 = 0$, are 1, 4 and 9, which three roots ought to be three of the roots of

of the given equation ; whereas only one of them is so. Indeed, the three roots of this reduced equation, are the squares of the roots of the given equation ; and not three of the roots of the given equation ; which the problem requires.

Suppose the given equation were

$$x^9 - px^6 + qx^3 - r = 0,$$

make $x^3 = z$, and you will have

$x^9 - px^6 + qx^3 - r = 0$, an equation of 9 dimensions, reduced to a cubic, by a simple cubic $x^3 = z$; which, by the lemma, you may say, ought to be an equation of $\frac{9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3} = 84$ dimensions. But this reduction has no relation to the lemma ; since the three roots of the reduced equation, are not what the lemma requires, three of the roots of the given equation ; but they are the cubes of the roots of the given equation. Such inattention to the conditions of the problem is inexcusable, and yet it is not here only that you are guilty of it. For the

Second of your mistakes, which I shall take notice of, proceeds from the same negligence ; " the first and most obvious way of reducing an equation, is, you say, by the simple equations out of which it is formed ;" (i. e.) supposing (x) to be the unknown quantity, and (α) one of the roots of the given equation ; if you divide the equation by $x - \alpha$, and make the quotient equal to (0) ; you have an equation, one degree lower than the given equation ; and whose roots will be likewise roots of the given equation : all which is certainly true, and as certainly nothing to the purpose : because the roots of the proposed equation, are not amongst the data of the problem.

To

To reduce an equation, having the roots given, is one problem.—To reduce an equation, having the equation only given, is another; and this is the subject of the lemma; which is not to shew, how equations may be reduced after they are solved; but how to reduce them in order to their solution, as I have observed to you before in my Reply. The other methods of reduction by division, which you mention, being only repetitions of this, I suppose nothing need be said about them, no more than about “the various ways,” which you do not mention, though you so confidently assert, “that there is not any one of them, in which the rule here delivered is true.”—After having seen two such gross instances of your inattention to the terms of the problem, the reader will not be surprized at your

Third mistake: and yet surely it is a strange mistake to assert, as you do, “that the problem before us is such, as admits of no determinate answer.” What you suspect, is true, that it was *Saunders*’s comment on *Des Cartes*, which suggested the problem, and the solution of it, to your young *Analyst*. Now *Des Cartes*’s problem is this; any biquadratic equation $x^4 + qx^3 + rx + s = 0$, being proposed: to find a quadratic, such as $x^2 + ex + f = 0$: the two roots of which shall be two of the roots of the given biquadratic; where the data are (q) (r) (s) , the coefficients of the biquadratic; and the qualita (e) and (f) , the coefficients of the quadratic; and the condition of the problem is, that the two roots of the quadratic, shall be two of the roots of the biquadratic. Now if you confine yourself to these terms, the problem is certainly

determinate ; it admits indeed of six answers, but it can have no more than six. But if you neglect these terms, and substitute new ones in the stead of them ; the problem will as certainly become indeterminate, and so will every problem that can be proposed, upon this or any other subject. Thus far *Des Cartes*. *Hudde* went farther. He has shewn, by different methods, how any proposed equation, $x^m - Ax^{m-1} + Bx^{m-2}, \&c. = 0$, of (m) dimensions, may be reduced to one of (n) dimensions ; such as $x^n - ax^{n-1} + bx^{n-2}, \&c. = 0$, where as before, the data are A, B, C , the coefficients of the first equation ; and the quæsita $a, b, c, \&c.$ the coefficients of the second ; and the condition of the problem is, that the (m) roots of the latter equation be (m) of the roots of the former ; (m) being supposed less than (n) ; upon which terms the problem is determinate : but if you neglect these, and introduce new terms in their place, you will without question render it indeterminate.

Now the difference between what *Hudde* has done, and the lemma in dispute, is this ; *Hudde* teaches us the solution of the problem : the lemma shews the dimensions of it. *Hudde* has given different methods, by which the *reducing* equation may be discovered : by the lemma you may see how high the *reducing* equation will rise, before you give yourself the trouble of finding it. If therefore *Hudde*'s be a determinate problem, mine, which is only to shew the dimensions of his, must be so too. And they both certainly are so, notwithstanding what you have so positively, and so rashly asserted to the contrary.

And it is upon the strength of these extraordinary blunders, that you break out into those rhetorical

exclamations; "a method of methods! a new discovery in mathematics! a proposition that ceases to be true, when it ceases to be useful; and a method so comprehensive, as to take in every unknown case, but so confined, as to belong to no known one." I have answered before, that the lemma takes in all cases, that fall within the terms of the problem, which is the subject of it; and I have above given you so many examples of the truth of it, that no more need be said on this head.

Then for your *new discovery*, it is an arrant quibble.

These methods of reduction, by the intervention of high equations, do certainly cease to be useful, when the roots of the equation to be reduced are given; but the assertion of the lemma does not therefore cease to be true; because the lemma asserts the necessity of these high *reducing* equations, not when the roots of the equation to be reduced are given, but when the equation itself only is given.

But your fourth mistake is, if possible, still more unaccountable than either of the three former; since every learner knows, that the dimensions of a rational quantity, are estimated by the index of the highest term; and yet you maintain, that an equation of this form $z^6 - pz^4 + qz^2 - r = 0$; is but a cubic. I had produced only one example in support of the lemma; it was *Des Cartes*' reduction of a biquadratic to a quadratic, by an equation of this form, $z^6 - pz^4 + qz^2 - r = 0$; which I asserted to be of 6 dimensions, the number required by the lemma, since $6 = 4 + 2$.

But this you *call in question*, because by making

$y=z^4$, the equation above mentioned may be reduced to a cubic, $y^3 - py^2 + qy - r = 0$. And it is not to be denied, that equations of this form, in reference to their solution, are often called cubics by the best writers; but you are the first that ever asserted them to be really so. *Saunders*on says, that an equation of this form $z^4 - pz^2 + q = 0$, is called a quadratic, because by making $z^2 = y$, it may be reduced to a quadratic $y^2 - py + q = 0$: but this very reason sufficiently indicates, that it is not in reality a quadratic, though in regard to its solution, it may be called so; and in p. 189, he says expressly, that this equation is, properly speaking, a biquadratic. But what is still more to the purpose, in his reduction of biquadratics, he calls an equation, of the form we are now disputing about, a bieubic: which, if it means any thing, must mean an equation of twice three, or 6 dimensions; for the same reason that a biquadratic, means an equation of twice two, or 4 dimensions. *Hudde* too, in his treatise on reductions, defines the dimensions of an equation, in which there are no radicals, by the index of the highest term; and in his 19th rule above cited, after having said, that if an equation of 6 dimensions, be reduced to three, the reducing equation will be of 20 dimensions; He adds, that it will be of this form, $z^{20} - pz^{18} + qx^{16}$, &c. $= 0$; where the index of the highest term being an even number, and the 2d, 4th, &c. alternate terms being wanting, if you make $y^4 = z$, you may reduce it to an equation of 10 dimensions $y^{16} - py^{12} + qy^8$, &c. $= 0$.

But to make this dispute as intelligible as may be, let us take an example such as $z^6 - 30z^4 +$

$z^6 - 30z^4 + 129z^2 - 100 = 0$. Now *Newton* says, "potest vero æquatio tot habere radices, quot sunt dimensiones, & non plures." But the equation $z^6 - 30z^4 + 129z^2 - 100 = 0$ has 6 roots, $+1, +2, +5, -1, -2, -5$, and therefore must be of six dimensions. But you are inclined to say, upon this occasion, what you would not say upon any other, that $+1$ and -1 (for example) are the same number: from whence, I suppose, you mean to conclude, that the 6 roots, which I have above enumerated, are in reality only three. To which it were a sufficient answer to say, that they are 6 roots according to Sir *Isaac Newton*'s definition; which is, "Radix vero est numerus, qui, in æquatione pro radice scriptus, efficiet terminos omnes se mutuo destruere;" and either of the six numbers above specified being substituted for (z) in the equation, will make the whole vanish.

But there is another way of considering this matter, which is not obnoxious to such pitiful caviling. Not only the equation $z^6 - 30z^4 + 129z^2 - 100 = 0$, has six roots; but the quantity $z^6 - 30z^4 + 129z^2 - 100$, (without supposing the whole equal to nothing) whatever be the value of (z) , has also six divisors $z - 1, z + 1, z - 2, z + 2, z - 5, z + 5$; and all these, multiplied together, will produce the very quantity $z^6 - 30z^4 + 129z^2 - 100$. But, if there be any sense in the words at all, a quantity which is the product of six simple factors, must be a quantity of six dimensions*: and though you are disposed to say, that $+5$ and -5 , for example, are but one number; you will hardly deny that $z + 5$ and $z - 5$ are two factors.

And *Newton*, Arith. Univ. p. 3 and 6,

And lastly, (FIG. 1.) if we make $z^6 - 30 z^4 + 129 z^2 - 100 = y$, and call AP, z, PM, y, MM , the locus of the equation, will cut the base in six points, *viz.* when $AP = z = +1$ or -1 , or $+2$ or -2 , or $+5$ or -5 ; in all which cases the ordinate $y = 0$. But a curve cannot cut a strait line in more points than the equation, by which it is defined, has dimensions. The equation therefore, concerning which we are disputing, has all the marks of an equation of six dimensions: 1st, Its highest term is of six Dimensions; 2dly, It has six roots; 3dly, it has six simple divisors; and 4thly, if you make $z^6 - 30 z^4 + 129 z^2 - 100 = y$, the locus will cut the base in six points.

5thly. You are mistaken in imagining, that *Saunderson's* reasoning is confined to *Des Cartes'* method, or others similar to it. "Dr. *Saunderson* has explained *Des Cartes'* method, as his practice is, clearly and diffusely; he has shewn that the root of the reducing equation, must have six values; because there may be six combinations of two of the roots of the biquadratic equation: and this it should seem, suggested the problem before us, and the solution of it to our young Analyst, who would have deserved praise, for pursuing his master's reasoning, and making it more general, if he had taken care fully to understand it; he would then have perceived, that it relates only to *Des Cartes'* method, or others similar to it; and that even here it does not shew the necessity of using an equation of 6 dimensions; since a cubic, whose three roots may have each two values, when the signs only are changed, will also serve the purpose."

Now

Now *Saunderson* assumes for the given biquadratic, $x^4 + qx^2 + rx + s = 0$; and for the quadratic sought, $x^2 + ex + f = 0$; and having found by his calculation, that the value of (e) is determined by an equation of 6 dimensions, and that of (e^2) by one of three only; he sets himself, in the next place, to account for this difference, and in order to this he takes an example $x^4 - 27x^2 - 14x + 120 = 0$, of which the roots are $+2, -3, -4$, and $+5$: each pair of which, as I have observed before, will give a quadratic equation, that will answer the conditions of the problem.

$$\begin{aligned} x^2 + x - 6 &= 0 \text{ are } +2 \text{ and } -3 \\ x^2 + 2x - 8 &= 0 \text{ are } +2 \text{ and } -4 \\ x^2 - 7x + 10 &= 0 \text{ which the roots } +2 \text{ and } +5 \\ x^2 + 7x + 12 &= 0 \text{ are } -3 \text{ and } -4 \\ x^2 - 2x - 15 &= 0 \text{ which } -3 \text{ and } +5 \\ x^2 - x - 20 &= 0 \text{ of which } -4 \text{ and } +5 \end{aligned}$$

From whence he concludes, that (e) the coefficient of the second term of the equation sought, $x^2 + ex + f = 0$, can have no fewer than six different significations; to wit, $\pm 1 \pm 2 \pm 7$; but its square (e^2) will have but three significations, viz. 1, 4, 49; because (1) is the square of both $+1$ and -1 ; 4 is the square of $+2$ and -2 ; and 49 is the square of both $+7$ and -7 ; the conclusion from which, by Sir Isaac Newton's rule, is, that the equation, which determines the value of (e) , must be of 6 dimensions, and the equation that finds (e^2) of only three.

The equations are $e^6 - 54e^4 + 249e^2 - 196 = 0$, and making $y = e^2$

$$y^3 - 54y^2 + 249y - 196 = 0;$$

where

where you may observe, that this cubic is so far from contradicting the lemma, that it is a confirmation of it, as both depend upon the same principle.

But you seem to make no distinction between the process, by which *Saunders*on finds the equations ; and his reasoning, by which he accounts for the dimensions of them. The process is from *Des Cartes* and his commentators ; the reasoning, by which he accounts for the dimensions, has no relation to the process ; but is drawn from the nature of the problem, and is the same with Sir *Isaac Newton*'s, in the rule abovementioned. There are various methods of discovering the values of (e) and (e^2) ; but whatever course you take, you will never find (e) by an equation of fewer than six dimensions ; nor (e^2) by one of fewer than three : *tres enim sunt casus diversi quæsitæ quantitatis* (e^2) ; *sex autem quantitatis* (e) ; *ab iisdem datis pendentes, & cædem argumentandi ratione determinandi*. It is a strange mistake therefore to imagine, as you do, that *Saunders*on's reasoning relates only to *Des Cartes*' method, or others similar to it ; since it depends entirely upon the nature of the problem, and has no relation at all to the process.

6thly. You are mistaken in asserting, that a cubic equation is sufficient for the reduction of a biquadratic to a quadratic.

To which I had answered before ; " as what the *Observer* says here, is not very intelligible, I shall endeavour to explain it. *Des Cartes* in order to reduce a biquadratic equation, to a quadratic, makes use of an equation of 6 dimensions of this form ;

" $x^6 - px^4 + qx^2 - r = 0$,

" suppose $\{ x^2 = z$, and you will have

" $\{ z^3 - pz^2 + qz - r = 0$, and this is
" the cubic equation, which the *Observer* says, is
" sufficient for the reduction of the biquadratic;
" but the *Observer* is mistaken: for the biquadra-
" tic is not reduced by means of this cubic alone,
" but by means of the

" cubic $\{ z^3 - pz^2 + qz - r = 0$,

" and quadratic $\{ x^2 = z$ both together; that is
by two equations, (one a cubic, and the other a
quadratic) with two unknown quantities; which
every one knows are equivalent to one equation
of 6 dimensions, with one unknown quantity:
and to this you have made no reply. But having
mistaken the fact, no wonder, that

7thly, You have been mistaken in accounting
for it.

The 3 roots of a cubic, you say, may have each
two values, when the signs only are changed:
which too I have taken notice of in the *Reply*.

" To understand this, the reader must be told,
" that *Saunders*on has proved, that the equation by
" which the biquadratic is reduced, must have six
" roots; which the *Observer* believed, and believing
" too, that the cubic alone was sufficient for the
" reduction, it behoved him to find in it six roots:
" and he finds them by *only* changing the signs;
" as if indeed the signs of an equation could be
" thus changed, and the equation still continue the
" same."

" The roots of the cubic equation,
" $x^3 - 2x^2 - 5x + 6 = 0$, are $+1, -2, +3$:
" change the signs of the 2d and 4th terms, and you
" will have $x^3 + 2x^2 - 5x - 6 = 0$; of which

“—1, + 2, —3, are the roots; and this is what
 “the *Observer* means, when he says, that the
 “roots of a cubic equation may have each two
 “values, when the signs *only* are changed; and
 “does he really think that we can thus change
 “the signs, without changing the equation? if he
 “does, he must think too, that a cubic equation
 “may have six roots; which indeed was what
 “he wanted to make out; though the impossibi-
 “lity of its having more than three, is demonstrated
 “in almost every book of algebra that is extant.”

8thly, You are mistaken in contradicting what I had asserted, that *Saunders*on has proved the necessity of an equation of 6 dimensions, in the reduction of a biquadratic to a quadratic. You say, *I am an apt scholar, if I can apply his proof to my purpose, and have learned from my master, more than my master knew.**

But you acknowledge, that *Saunders*on has shewn that the root of the reducing equation, must have six values.† You mean to say, what *Saunders*on says, that the unknown quantity of the reducing equation, must have 6 values; or which is the same, that the equation must have 6 roots. But the roots of an equation can not exceed the number of its dimensions: ‡ therefore, by your own confession, *Saunders*on has proved, that the reducing equation must be of 6 dimensions.

Your 8th mistake is in the conclusion.

If by the number of dimensions which the lemma requires, I mean the greatest number of dimensions, which the reducing equation in *any case can have*; then you say, my answer to the problem

* *Def. p. 10.* † *Obs. p. 16.* ‡ *Newt. Ari. Univ.*

lem cannot as far as you know, be demonstrated to be false. But indeed it can, and it is amazing you should not know it.

Change *greatest into least*, and you will be right : for an awkward mathematician by accident, or a master by design, may often find an equation of more dimensions, that will answer the purpose. But all the art of man can never solve the problem, by an equation of fewer dimensions, than are required by the lemma. No maximum can be assigned to the dimensions of an equation, by which any problem is to be solved ; because there are an infinity of equations, that will solve every problem. You have a

9th Mistake in the same conclusion. " He that would make the solution of an high equation easier, must shew us how to reduce it by another, which has fewer roots."

Now this is absolutely impossible, since there is no general rule by which an equation can be reduced lower, without solving an equation, of at least as many dimensions as that which is to be reduced ; and all the common solutions contradict what you have here asserted. A quadratic is solved, not by reducing it to a simple equation, but to a simple quadratic. A cubic is turned into an equation of 6 dimensions, in order to its solution. A biquadratic is indeed reduced to a quadratic, but not without the intervention of an equation of 6 dimensions.

And so much for the first lemma ; which I have examined fully, because you have rested the whole Dispute upon it.* " If he only gives us a clear demonstration, drawn out in form and method of

* *Def. p. 7.*

" this proposition ; the equation which reduces
 " another to (m) Powers, has for its roots, all the
 " combinations of (m) roots of the former ; I ac-
 " accept it as a full answer to all my objections."

I have given you the demonstration, which you wanted. And as this specimen will be enough to satisfy the curiosity of the reader, and enable him to judge of the merit of the rest of the performance, I shall be very short in what follows.

LEMMA II.

LET (a) be the sum of the roots of any given equation ; (b) the sum of their squares ; (c) the sum of their cubes, &c. and I have these three problems.

1st, To find any term, whose place in the series, $a, b, c, d, \&c.$, is denoted by (n).

2dly, To find the sum of the (n) first terms.

3dly, To find the sum of all the terms, ad infinitum.

The 1st of these problems is the subject of the lemma.

Ex. (1.)

Let the equation be $x^2 - px + 1 = 0$, then by Sir Isaac Newton's rule, we have

Tab. 1.

Tab. 2.

$$\begin{aligned}
 a &= p &= p \\
 b &= pa - 2 = p^2 - 2 \\
 c &= pb - a = p^3 - 3p \\
 d &= pc - b = p^4 - 4p^2 + 2 \\
 e &= pd - c = p^5 - 5p^3 + 5p \\
 f &= pe - d = p^6 - 6p^4 + 9p^2 - 2.
 \end{aligned}$$

The 2d table is easily made from the first,

thus

$$\text{thus } b = pa - 2$$

$$\text{but } pa = p^2 \text{ therefore } b = p^2 - 2$$

$$\text{again } pb = p^3 - 2p$$

$$a = \dots p$$

$$c = pb - a = p^3 - 3p$$

$$pc = p^4 - 3p^2$$

$$b = p^2 - 2$$

$$d = pc - b = p^4 - 4p^2 + 2, \text{ and so of the rest.}$$

In general, let α and β be two of the roots of the proposed equation $x^n - px + 1 = 0$, and I say that $\alpha^n + \beta^n = p^n - np^{n-2} + n \cdot \frac{n-1}{2} p^{n-4} - n \cdot \frac{n-2}{2} \cdot \frac{n-4}{3} p$, &c.

This series may be compared with the tables, by substituting 1, 2, 3, 4, &c. successively for (n); by which substitution, you will find the same values of $a, b, c, \text{ &c.}$ as in the tables. The general demonstration of it is become unnecessary, because you no longer dispute it.* Instead of which therefore, I will give you an instance or two of its use.

FIG. 2. Let C be the center, and $AP = 2$, a diameter of the circle ABC : $AB, BD, DE, \text{ &c.}$ equal arcs, $PB = a = p$, $PD = b$, $PE = c$, &c. the chords of the supplements of the arcs, to a semicircle, then it follows from Vieta's rule, that

$$PB = a = p$$

$$PD = b = pa - 2 = p^2 - 2$$

$$PE = c = pb - a = p^3 - 3p$$

$$PF = e = pc - b = p^4 - 4p^2 + 2$$

$$PG = f = pc - d = p^5 - 5p^3 + 5p, \text{ &c.} \dagger$$

Now comparing this third table with the first, you will find, that Vieta's rule for finding the chords of the supplements of the multiple arcs, is

* *Dif. p. 13.* + See *De L'Hospital Con. Sect. p. 414.*

the same as Sir *Isaac Newton's*, for finding the several powers of the roots of this quadratic equation, $x^2 - px + 1 = 0$.

And if you consult the 29th problem of the *Arith. Univ.* p. 158.

"Datum angulum per datum numerum multiplicare," you will find (by making $r=1$) *Newton's* equations, and those in the 4th table are exactly the same. Whatever injury therefore I may have done Sir *Isaac Newton's* rule, by *changing the expression* of it, the same injury has been done by Sir *Isaac* himself to *Vietta's* rule.

But from the tables abovementioned, we may draw the following corollaries.

1st, If the arc AG , be to the arc AB , as (n) to (1) , then by *Newton's* rule

$\alpha'' + \beta'' = PG$. But α and β , the roots of the equation $x^2 - px + 1 = 0$. are $\frac{1}{2}p + \sqrt{\frac{1}{4}p^2 - 1}$, and $\frac{1}{2}p - \sqrt{\frac{1}{4}p^2 - 1}$, and consequently

$$PG = \frac{1}{2}p + \sqrt{\frac{1}{4}p^2 - 1} + \frac{1}{2}p - \sqrt{\frac{1}{4}p^2 - 1}.*$$

And for the same reason,

Cor. 2. $PG = \alpha'' + \beta'' = p'' - np''^{-1} + n \cdot \frac{n-3}{2} p''^{-2} - n \cdot \frac{n-4}{2} \cdot \frac{n-5}{3} \cdot p''^{-4}$, &c. +

Cor. 3. An equation of this form, $p'' - np''^{-1} + n \cdot \frac{n-3}{2} \cdot p''^{-2}$, &c. = l , being given: if (1) be not greater than (2) , the roots of it may be found by a table of sines.

Ex. Let $l = AP = 2$ and $n = 6$, then the equation will be $p^6 - 6p^4 + 9p^2 - 2 = + 2$, or $p^6 - 6p^4 + 9p^2 - 4 = 0$.

FIG.

* See *Mac Laurin's Fluxions*, p. 619. and *De Moivre Mif. Anal.* p. 1. & 13. † See *De L'Hopital*, p. 450. pr. 9.

FIG. 3. Divide the circumference into 6 equal parts $AB, BD, DE, \&c.$ and draw the chords $PB, PD, PE, \&c.$ and the roots will be $PB, PB, -PF, -PF, +PA, -PA;$ or since $PB = PF = 1,$ and $PA = 2,$ the roots are in this case $+1, +1, -1, -1, +2, -2.$ and $\frac{p-1}{p-1} \times \frac{p-1}{p-1} \times \frac{p+1}{p+1} \times \frac{p+1}{p+1} \times \frac{p-2}{p-2} \times \frac{p+2}{p+2} = p^6 - 6p^4 + 9p^2 - 4,$ &c.*

Ex. 2.

Let the equation be $x^6 - px + q = 0,$ and then by *Newton's rule*,

$$a = p$$

$$b = pa - 2q = p^2 - 2q$$

$$c = pb - qa = p^3 - 3qp$$

$$d = pc - qb = p^4 - 4qp^2 + 2q^2$$

$$e = pd - qc = p^5 - 5qp^3 + 5q^2p$$

$$f = pe - qd = p^6 - 6qp^4 + 9q^2p^2 - 2q^3.$$

and in general, supposing α and β the roots of the equation, $x^6 - px + q = 0,$

$$\alpha^n + \beta^n = p^n - np^{n-2}q + n \cdot \frac{n-3}{2} p^{n-4} q^2, \&c.$$

COR. Hence we may find the divisors of $1 - \Pi^6,$ which is Mr. *Cotes's Theorem.*†

For Example, take $1 - \Pi^6,$ and

Let $x^6 - 1 + \Pi^6 x + \Pi^2 = 0,$ of which the roots are 1 and $\Pi^2,$ and comparing this equation with $x^6 - px + q = 0,$ we have $\alpha = 1, \beta = \Pi^2, p = 1 + \Pi^4,$ and $q = \Pi^2;$ and consequently $\alpha^6 +$

$$\beta^6 = 1 + \Pi^12 = \overline{1 + \Pi^2}^6 - 6 \cdot \overline{1 + \Pi^2}^4 \Pi^2 + 9.$$

$$\overline{1 + \Pi^2}^4 \Pi^4 - 2 \cdot \Pi^6 \text{ and } 1 - 2 \Pi^6 + \Pi^{12} =$$

$$\overline{1 + \Pi^2}^6 - 6 \cdot \overline{1 + \Pi^2}^4 \Pi^2 + 9 \cdot \overline{1 + \Pi^2}^2 \Pi^4 - 4 \cdot \Pi^6.$$

* See the Authors above mentioned. † Vid. Hor. Mens.

$$\Pi^6 \text{ and } \frac{1 - 2 \Pi^6 + \Pi^{12}}{\Pi^6} \Rightarrow \frac{1 + \Pi^3}{\Pi^6} - 6 \cdot \frac{1 + \Pi^3}{\Pi^6}$$

$$+ 9 \cdot \frac{1 + \Pi^3}{\Pi^6} - 4.$$

$$\text{Let } z = \frac{1 + \Pi^3}{\Pi} \text{ and suppose}$$

$z^6 - 6 z^4 + 9 z^2 - 4 = 0$ and dividing the circle into six equal parts as before, the roots of this equation will be $+PD, +PD, -PF, -PF, +PA$ and $-PA$, or since $PA = z$, and $PD = PF = 1$. The roots will be $+1, +1, -1, -1, +2, -2$, and consequently

$$z - 1. z - 1. z + 1. z + 1. z + 2. z - 2.$$

$$= z^6 - 6 z^4 + 9 z^2 - 4 = \frac{1 - 2 \Pi^6 + \Pi^{12}}{\Pi^6}$$

$$\text{or } z - 1. z + 1. z + z. z - 2 = \frac{1 - 2 \Pi^6 + \Pi^{12}}{\Pi^6} \text{ whatever be the value of } z,$$

and without supposing

$$z^6 - 6 z^4 + 9 z^2 - 4 = a. \text{ But } z = \frac{1 + \Pi^3}{\Pi} \text{ and}$$

$$\text{consequently } z - 1 = \frac{1 + \Pi^3}{\Pi} - 1 = \frac{1 - \Pi + \Pi^3}{\Pi}:$$

$$\text{so } z + 1 = \frac{1 + \Pi + \Pi^3}{\Pi}, \text{ &c. } z + 2 = \frac{1 + 2 \Pi + \Pi^3}{\Pi}$$

$$= \frac{1 + \Pi}{\Pi} \text{ and } z - 2 = \frac{1 - \Pi}{\Pi} \text{ and consequently}$$

$$\frac{1-\pi+\pi^2}{\pi^2} \times \frac{1+\pi+\pi^2}{\pi^2} \times \frac{1+\pi}{\pi} \times \frac{1-\pi}{\pi} =$$

$$\frac{1-2\pi^6+\pi^{12}}{\pi^6}, \text{ and } \frac{1-\pi+\pi^2}{1+\pi+\pi^2} \times \frac{1+\pi+\pi^2}{1-\pi} \times$$

$$\frac{1+\pi}{1+\pi} \times \frac{1-\pi}{1-\pi} = 1 - \pi^6.$$

And now, considering that the two examples, which I have here given, are almost the simplest that belong to the rule; and that the corollaries, which I have drawn from them, are by all writers reckoned amongst the most extraordinary inventions of the last century: If I was to treat with some sort of contempt, what you have falsely said of the uselessness of this problem, I don't see how you could reasonably be surprized, or angry at me.

LEM. 2. COR. I.

To find the (n) first terms of the series $a+b+c+d, \&c.$

Let the equation be $x^2 - px + 1 = 0$.

Now the sum of the (n) first terms is (see the 1st table) $(p+p^2+p^3+p^4, \&c.$ continued to (n) terms $= \frac{p-p^{n+1}}{1-p} - (2+3p+4p^2+5p^3, \&c.$ to

$$n-1 \text{ terms } =) \frac{2-p-n+1 \cdot p^{n-1} + np^n}{1-p} + \&c. \text{ therefore } a+b+c+d, \&c. (n) \text{ terms } = \frac{p-p^{n+1}}{1-p} - \frac{2-p-n+1 \cdot p^{n-1} + np^n}{1-p} \&c.$$

Ex. Let the equation be $x^2 - x + 1 = 0$, and let it be required to find the four first terms; in

F which

* Vide *De Moivre, Mis. Anal.* p. 167.

which case $p=1$ and $n=4$; and consequently

$$\frac{p-p^{n+1}}{1-p} = \frac{p-p^5}{1-p} \text{ which (when } p=1 \text{) is (4).}$$

And this is what you have represented as so great a mystery, since when $p=1$, then

$$\frac{p-p^5}{1-p} = \frac{1-1}{1-1} = \frac{0}{0} = 4.$$

Your misrepresentation had raised a prejudice against me: and therefore I thought myself obliged to support what I had said, by authority; and I did it by the authorities of *De Moivre*, *Mac Laurin*, *Saunderson* and *Nicholas Bernoulli*; who all agree that the value of the fraction $\frac{p-p^5}{1-p}$ (when $p=1$) is 4.

And these writers you acknowledge to be accurate; but you add that "every one of them is "careful to shew, that he is speaking only of the "limit of the ratios of two variable quantities, "supposed to decrease *in infinitum*. And had the "Professor applied their doctrine in a similar manner, the objection to his solution would have been "only this, that he introduces fluxions or ultimate "ratios into a problem, which may be solved by "the easiest operations of arithmetic, with a tenth "part of the figures; and by most people, we may "suppose, in an hundredth part of the time."*

But this is manifestly misrepresented: *Mac Laurin* has two methods of finding the value of the fraction; one by common division, and the other by fluxions: the former I had quoted; the latter you quote, as if it was part of the former, though it has no relation to it. But *Mac Laurin* uses the word

vanish

* *Def. p: 33.*

vanish, which is enough for you to cavil upon. *Nic. Bernoulli* finds the value of the fractions in question, by fluxions; but Mr. *De Moivre* is so far from doing it by fluxions himself, that having explained *Bernoulli's* method; He adds, “ Quam vero regula clarissimi viri sit optima, pri-“ maque, quod sciam, quæ ad hanc rem tradita fu-“ erit; attamen in casu proposito eandem conclu-“ sionem aliquanto expeditius obtinere licebit:” and then he shews, how to find the value of these fractions, by common arithmetic: and, what is more, in stating the problem, he has not so much as a fluxional expression. It is therefore absolutely false to say, as you do, that every one of these writers refer this matter to ultimate ratios and fluxions.

Then, as to what you say to the discredit of the problem, I am content to leave it to the authority of those authors, whom I quoted in the *Reply* from *De Moivre*, *Mis. Anal.* rather than fill my book with a long calculation to shew the falsity of your assertion.

LEMMA 2. COR. 3.

To find the sum of the series $a+b+c+d+e$, &c. ad infinitum,

Let the equation be $x^n - px^{n-1} + qx^{n-2} + \dots = 0$, and the sum required will be $\frac{p-2q+3r-4s+\dots}{1-p+q-r+s+\dots}$, &c.

*It should have been added you say, if each root of the equation be less than 1 and greater than - 1;** and, in your defence, you have asserted roundly, that the problem is in all cases either useless or uncertain. “ For either we know all the roots of the equa-

* *Obs. p. 12.*

“ equation to which we apply it, or not. If we
 “ know them, and they are within the limits as-
 “ signed, the rule is useless. If not, it is worse
 “ than useless ; it leads us into error. For there
 “ is infinitely greater variety, where it is false, than
 “ true.” But it is your misfortune to look for eve-
 ry thing in the wrong place ; and here you are
 diving to the bottom of your *deep well*, in quest of
 the *hidden* radical *treasures*, neglecting the coeffi-
 cients, which lie *upon the surface*, *OPEN before*
you.

The roots are out of the question ; there is no mention made of them in either the lemma or the corollaries : but the series is a *given* series. As many terms as you please may be found by the problem in the lemma, or by Sir *Isaac Newton*’s rule, and all that is left to the sagacity of the reader, is, to see whether the series, which he has before his own eyes, converges or diverges.

Ex. Let $x^6 - 4x^5 - 2x^4 + 20x^3 - 11x^2 - 16x + 12 = 0$; then by your own calculation $a = 4$, $b = 20$, $c = 28$, $d = 116$, $e = 244$, &c. * and the series is $4 + 20 + 28 + 116 + 244$, &c. and, I suppose, it is no great compliment to the skill of my reader, if I take it for granted, that he will not attempt to find the sum of a series, which so plainly diverges.

Again, let $x^3 - \frac{1}{3}x^2 + \frac{1}{2}x - \frac{1}{6} = 0$,
 and you will find $a = \frac{1}{3}$, $b = -\frac{8}{9}$, $c = \frac{1}{27}$, $d = \frac{83}{162}$, $e = \frac{1}{243}$, &c. and the series is

$\frac{1}{3} - \frac{8}{9} + \frac{1}{27} - \frac{83}{162} + \frac{1}{243}$, &c. which converges ; and the sum is $-\frac{1}{6}$.

I referred you, in my reply, to *De Moivre* and *Stirling*. You complain that I do not refer you to the

* *Obs.* p. 14.

the particular passages ; but who could have imagined, that it was necessary ? There are but 33 propositions in *Stirling's* book : you might have examined them all in a few minutes. *De Moivre* has an index—in which there are but 25 articles : it could have given you no great trouble to have run your eye over this index. No doubt, but you did examine it carefully. But the truth is, that though you speak so familiarly of that excellent Analyst, you are no more acquainted with his works, than with those of *Des Cartes* and his commentators ; and though you so deservedly commend Sir *Isaac Newton's* rule, it is plain you do not know it again, when you see it expressed in different words.

Sir *Isaac Newton's* rule is, that the series $a+b+c+d+e$, &c. is what Mr. *De Moivre* calls a *series recurrens*; or as Mr. *Stirling* expresses it, without a technical word, “ series est, in quâ, relatio “ terminorum est constans.” I told you, that *De Moivre* had many pages on the subject of the lemma ; I might have added, half his book : he has a whole chapter on the subject of this corollary, *De summis serierum recurrentium* : and it is the first example of the 15th proposition of *Stirling's* *summatio serierum*. But for your limit ; “ each of the roots, you say, must be greater than (-1) and less than (1).” By which, I dare say, you mean no more than this, that if either of the roots be equal to, or greater than (1), the series $a+b+c+d$ &c. will not be finite; which is true. But your expression has a great deal of affectation in it; and, what is worse, in consequence of this affectation, though what you mean to say, is true ; yet what you do say, is false.

The

The roots of the equation in the last example, are $\frac{1}{2}$, $+\sqrt{-\frac{1}{2}}$, and $-\sqrt{-\frac{1}{2}}$; and I believe you will not venture to say, that either of the two last is greater than -1 , and less than 1 .

And now, to conclude. I have given you the demonstration of the 1st lemma, which you required; and upon which you have rested the whole dispute; and I have explained the problems in the 2d lemma, by such examples and corollaries, as will no longer leave you room to doubt of the use, or the truth of them. Your objections to the other parts of my book were of no importance, even in your own eyes; and there is evidently no longer any dispute between us about them.

And thus much in my own defence: on the other hand, I have complained again, and must for ever complain of your cruelty; and from what I have said upon this head, every impartial reader will, I hope, be convinced, that I have but too much reason to complain of the manner, in which the *Observations* are written; as well as the occasion, on which they were published.

Then I have retorted the charge of obscurity upon yourself, and have given one example of it, which may serve instead of many. I could have produced many more even out of your *Observations*, and as for your *Defence*—clouds and darkness are round about you from the beginning to the end of it. Lastly, I have examined minutely what you have said upon the first lemma, &c. “and from these “ observations on the prelude, it will not be difficult to conjecture, what entertainment is prepared for us in the other parts of the work.”†

Perhaps

† *Obf.* p. 20.

Perhaps I may find a future opportunity of examining some of them with the same attention. In the mean time, the reader seeing, from hence, how you reason yourself; may judge how able you are to *distinguish good reasoning from bad, in others; which, it seems, is now all that you pretend to in this science.**

* *Def.* p. 38.

I am, &c.

E. W.



Fig. 1.

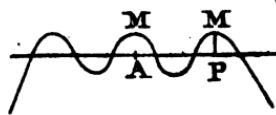


Fig. 2.

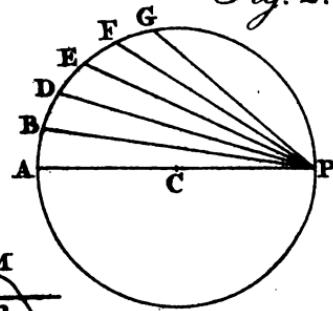


Fig. 3.

